

Judgments, contents and their representations

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Abstract

1 Introduction

Among the many lessons that my thesis supervisor Göran Sundholm taught me, one of the most important ones is that logic is essentially about *inferences*, i.e. moves from knowledge of the premises to knowledge of the conclusion, and thus not about *consequences*, i.e. abstract relations between contents.¹ For this reason, Sundholm maintains that the key elements in logic must be judgments, not contents or propositions. Frege is a great source of inspiration for his views of logic, and he has indeed advocated “logical atavism” ([19]): we should turn to Frege for inspiration, not to our immediate predecessors, i.e. those belonging to the meta-mathematical tradition consolidated after the 1930’s. According to Sundholm, logic was fundamentally misled after Frege, in no small measure due to the reduction of judgments to contents and of inferences to consequences.

In this paper, I suggest that this fact may partially be Frege’s own fault, and this for two reasons related to his choice of notation in the *Begriffsschrift*. Firstly (and less importantly), his notation to denote assertion, the famous judgment stroke, is cumbersome and extremely inconvenient for type-setting. Secondly (and more importantly), while Frege deems judgments to be the basic elements of logic, the notation for judgments is not a primitive element of his notational system: it is formed by the application of the functor judgment stroke “|” to a capital letter preceded by the horizontal dash representing judgeable content “-A”, yielding “⊢ A”. This notational choice suggests that contents, not judgments, are the primary entities for logic.

I first survey Sundholm’s conception of logic (which he views as essentially Fregean) as an *organon* for the production of new knowledge, where judgments and inferences occupy the central stage. I then discuss the tension that emerges from Frege’s emphasis on judgment and his notational choices. Thirdly, I briefly examine what happened to the notions of inference and judgment in the developments after Frege, and conjecture that his notational choices may have played a role in the exile of judgment from logic. Fourthly, I contrast Frege’s representation of judgments as the combination of content and judgment stroke with medieval representations of judgments and contents. Judgments – *propositiones*,

¹Whether I have come to agree completely with his distinctive view of logic is a different matter (which shall be further addressed in this paper); but his epistemological approach to logic has always been stimulating and thought-provoking. Indeed, I consider it to have been an essential element of my formative years as his PhD student.

assertions – were also the (sentential) primitive elements for medieval logicians, and many of them (such as Abelard and Buridan) recognized the so-called Frege Point in some sense or another. However, in their logical language – the regimented form of Latin that was medieval academic Latin – judgments receive a more basic syntactic representation than contents. I shall focus on Abelard, who arguably “discovered” propositionality more than seven centuries before Frege, and thus shares a significant portion of common ground with Frege. In conclusion, these considerations shall allow me to reflect on the importance of notational devices mirroring the conceptual framework underlying a logical system, and to question what the almost unanimous attribution of a canonical form to contents instead of to judgments may mean for the status of the epistemological conception of logic.

2 Sundholm on logic: judgment, inference and knowledge

As is well known, Sundholm belongs to the “constructivist school” in logic, the school having its roots in the Dutch intuitionistic movement, which was then further developed in particular by Swedish logicians such as Dag Prawitz and Per Martin-Löf, but also by Michael Dummett. Roughly put, this school is characterized by an emphasis on the epistemological import of logic and on the grounds for the knowledge obtained (not exclusively, but primarily) through logic. The correctness of a judgment (for example, that a certain inference is valid) must be solidly grounded: one must actually present that in virtue of which the judgment is correct, its ground. In the case of an inference, one must present the *proof* leading from premises to conclusion; more generally, one must display either the procedure – the appropriate act – or the traces thereof – the construction – that has allowed the agent to produce the piece of knowledge expressed in a judgment.

Sundholm has advocated his epistemological view of logic in several of his writings. His main motto may be said to be: “*Logic is an epistemological tool for obtaining new knowledge from known premises*” ([18], p.31). In particular, Sundholm views Frege as a kindred spirit, and interprets Frege’s much-debated contention that a valid inference must have true – in fact, *known* – premises in the light of this epistemological vision of logic ([20], p. 572). If the premises are not known, then the conclusion will not be properly grounded, and thus no new knowledge will be obtained by means of the passage from premises to conclusion, even if the passage itself is warranted:

*“Nothing at all can be inferred from false premises. A mere thought, that has not been recognized as true, cannot be a premise. Mere hypotheses cannot be premises.”*²

Sundholm deplores the almost total absence of this epistemological component in most of 20th century work in logic, in particular under the influence of

²Letter to Jourdain, [9], p.H8. English translation from [18]. As is well-known (and aptly discussed by Bell in his [3], III), this view poses serious problems concerning the possibility of proofs by absurdity, as in such proofs the initial premise is not seriously asserted but rather posited precisely so that a contradiction can be derived from it, thus proving its falsity.

Hilbert and Tarski, but with precedents already in Bolzano and even in the medieval notion of *consequentia* ([18]). He locates the ultimate “fall” in the period immediately after Frege, in particular with Whitehead and Russell’s *Principia Mathematica*, which was then further consolidated both in Hilbert’s formalistic program and in Tarski’s model-theoretic tradition. Indeed, in the formalistic program, there is an outright conflation of judgments and contents (see [17]).³ Moreover, Tarski’s wide-ranging influence was also particularly nefast: Tarski made the supreme *faux-pas* of reducing the validity of an inference to the holding of a consequence. There is, however, a fundamental difference between inferences and consequences: (valid) inferences fulfill the epistemological role of producing new knowledge, but consequences simply cannot do that. Therefore, if consequences become the chief object of logic, then logic can no longer have the epistemological import that Sundholm and Frege assign to it. As Sundholm puts it, logic in the 19th and 20th century gradually abandoned its status of “epistemo-logic” and became increasingly “onto-logic” [17].

Let us thus take a closer look at the crucial differences between inference and consequence in order to understand why consequences cannot produce new knowledge. Here is how Sundholm ([18], p.30) defines these two concepts:

- A consequence is a relation between propositions that may hold;
- An inference is an act of passage from judgment(s) to judgment that may be valid.

The two most visible differences are thus that (i) a consequence is an abstract *relation*, floating in a Platonic realm somewhere, whereas an inference is an *act* performed by an agent; (ii) the relata of a consequence are two sets (the set of premises and the set of conclusion(s)) of *propositions*, whereas the relata of an inference are two sets of *judgments*.⁴ Judgments are themselves acts performed by an agent, so an inference is an act acting upon acts in such a way that it “*effects a passage from known judgments to a novel judgment that becomes known in virtue of the inference in question.*” ([18], p.27). Again, this conception of inference is essentially the one held by Frege, which Sundholm fully endorses:

*“An inference [...] is an act of judgment that is drawn according to logical laws from judgments previously made. Each premise is a certain proposition which has been recognized as true, and also in the conclusion-judgment a certain proposition is recognized as true.”*⁵

Now, given that producing new knowledge is obviously an act conducted by an agent, the mere fact that an inference is an act and a consequence is not (it is a relation, independent of any agent’s doings) immediately implies that consequences can obviously not produce anything, and thus a fortiori cannot

³In the proof-theoretic tradition stemming from the work of Gentzen the significance of inferences for logic is still preserved, even though Gentzen’s sequent calculus is an uninterpreted calculus. See [20].

⁴This characterization of consequences is essentially that of Bolzano. In [21], for example, consequences do not have this Platonic ontological status, but this dissimilarity is immaterial for the present purposes.

⁵[7], p.387; english translation from [18]

produce knowledge: a product is by definition the outcome of an act of producing. But naturally, the proponents of the reduction of (valid) inferences to consequences (that hold) do not maintain that inferences and consequences are entirely (conceptually) equivalent; rather, they maintain that each valid inference corresponds to a consequence that holds, *mutatis mutandis*. In other words, given a certain inference, in order to evaluate its validity it is sufficient to convert it appropriately into a consequence (premises and conclusion become propositions; the act of inferring becomes a relation) and then evaluate whether the consequence holds. With this reduction, logic is then free to focus on the study of consequences, as presumably knowledge of the validity of inferences would simply follow from it. The underlying assumption is that of extensional equivalence between consequences that hold and, *mutatis mutandis*, valid inferences.

What are the necessary and sufficient criteria for a putative consequence to hold? This is of course one of the main debates in recent philosophy of logic. After decades of Tarskian, model-theoretic homogeny, the debate was ignited anew with the publication of Etchemendy's *The Concept of Logical Consequence* ([5]). I have argued elsewhere ([4]), and following the medieval notion of *consequentia*, that the fundamental aspect of the notion of consequence is the incompatibility of the truth of the premises with the falsity of the conclusion, i.e. what Etchemendy calls the representational notion of (logical) consequence, rather than the property of substitutivity without loss of validity of a consequence's non-logical terms, i.e. what Etchemendy calls the interpretational notion of (logical) consequence. But this dispute is immaterial for the present purposes, as in both cases a consequence may hold independent of an agent's (grounded) assent to it. That is to say: a given consequence C may hold even if no agent has been through the process of proving that it indeed holds, or in any case not the particular agent performing a (putative) inference I whose validity is presumed to be grounded on the fact that the corresponding consequence C does hold.

That consequences thus described are not adequate tools for the production of new knowledge is made patent by the phenomenon of "blind knowledge". The point is that, if knowledge is indeed justified true belief, then a consequence fails to produce new knowledge because, even if the belief thereby produced happens to be a true one by sheer chance, it is still not properly justified. "Blind knowledge" is thus *unjustified* true belief, and hence it is not knowledge properly speaking (just as a false friend is not a friend properly speaking). Suppose I am asked about the number of windows in the Empire State Building; I haven't got a clue, but nevertheless I utter "1.347", just as I could have uttered any other number (but somehow "something" tells me this is the right number). Now, suppose that coincidentally the number of windows in the Empire State Building is indeed 1.347; well, it turns out that I have made a judgment that happens to be true, even though I had no grounds for it. Clearly, it would be very strange to count this judgment of mine as real knowledge – even though it was (it expressed) a true belief – precisely because it was not justified. The issue is now that if valid inferences are defined as those corresponding to a consequence that holds, many of those purported inferences will be "blind inferences": if all it takes for a consequence to hold (and thus for the corresponding inference to be valid) is the impossibility of its antecedent(s) being true while its consequent is false, then an agent need not be in possession of the grounds for the legitimate move from premises to conclusions in order for her judgment of the validity of an

inference to be correct. It will be sufficient that the corresponding consequence holds in some abstract, objective way, even if she has no knowledge of this fact.

An example may help illustrate the point. We now know that Fermat's Last Theorem (FLT) is a consequence of the basic axioms of number theory: because we have a proof of FLT, we know that it is impossible for the basic axioms of number theory to be the case while FLT is not the case. Now, before Wiles' proof of the theorem, this consequence (basic axioms of number theory \Rightarrow FLT) obviously already held, even though we did not know it to be the case then. But if all it takes for an inference to be valid is that it corresponds to a consequence that holds, then somebody claiming to effect the inference from the basic axioms of number theory to the truth of FLT *prior* to the discovery of the proof of FLT would, according to this criterion, be effecting a valid inference and thus producing new knowledge. But this is of course absurd: our true belief in FLT only became justified once a *proof* of FLT was found.⁶ Before that, such a purported "inference" was merely a "blind inference". Indeed, this very proof is precisely the (valid) inference from the basic axioms of number theory to FLT, and only thereby do we really acquire new knowledge, i.e. new *justified* true belief.

Hence, as Sundholm has been arguing relentlessly, if logic is indeed essentially concerned with the production of new knowledge, its basic elements can neither be consequences nor inferences whose validity depend solely on the holding of the corresponding consequences. A stricter criterion for the validity of inferences is required in order to avoid the undesired consequences of "blind knowledge", one that emphasizes the actual epistemic process of arriving at the conclusion assuming knowledge of the premises, e.g. (as described by Frege) judgments "*drawn according to logical laws*" (see passage above). Now, this epistemic process is entirely constituted of acts of judgments: an agent first judges the premises to be true, and then she judges the move from the premises to the conclusion to be a valid, grounded inference (and thus in fact undertakes this move); she is thereby entitled to real knowledge of the conclusion, i.e. she has grounds to assent to the conclusion.

Moreover, if inferences thus construed are viewed as the primary elements of logic, consequences can in turn be approached on the basis of inferences. Once a putative inference is established as a valid inference (by means of a proof), then we know that it corresponds to a valid consequence, as clearly it is impossible for the contents of the judgments corresponding to the premises to be the case while the content of the judgment corresponding to the conclusion is not the case.⁷ As for the (presumed) consequences that hold "out there", but whose corresponding inferences have not (yet) been established as valid, their role in and for logic would be completely idle, as they cannot be involved in the production of new knowledge. So one need not deny the existence of consequences that do not (yet) correspond to valid inferences (implying thus that the class of valid inferences is strictly contained in the class of consequences that hold, *mutatis mutandis*); but they are simply of no interest to logic construed as an epistemological tool.

Of course, it may still be argued that logic is *not* essentially concerned with the production of new knowledge, in which case a completely different picture

⁶Interestingly, before the proof was found, FLT was widely but merely intuitively thought to be true indeed. But obviously, at that point it was also widely thought that the (true) belief in FLT was not justified.

⁷Assuming, of course, the soundness of the deductive system being used.

of what logic is about must be provided. But if one subscribes to this view of logic, then clearly one ought to take judgments, not contents/propositions, as its basic elements. This is indeed Frege’s explicit position; implicitly, however, given his notational choices in the *Begriffsschrift*, he seems to take contents as primitive elements, thus threatening the dissemination of his own vision of logic, as I shall argue now.

3 Frege on judgment, and his notational “slip”

In this section, I argue that, while from the point of view of Frege’s philosophies of mind and language, a thought (content, proposition) may be more basic than a judgment, for logic (as Frege understands it) judgments must be treated as basic elements (as argued in the previous section), but that this fact is not properly represented in his notational choices in the *Begriffsschrift*.

Frege presented his views on content, truth and judgment in several of his writings, most notably in *Über Sinn und Bedeutung* and in *Der Gedanke*. These views are at the crossroad between epistemology, philosophy of mind and philosophy of language, and have thus a series of important implications that will be of no concern for us here. For our purposes, a characterization of the gist of his notion of judgment should suffice.⁸ Here are two passages where Frege comes very near to providing a full-fledged definition of the concept:

*“A judgment for me is not the mere comprehension of a thought, but the acknowledgment of its truth.”*⁹

*“Judging can be regarded as advancing from a thought to its truth value. Naturally, this cannot be a definition. Judging is something quite of its own kind and incomparable.”*¹⁰

The idea is the following: just as a term has a *Sinn* and a *Bedeutung* (although not all term have a *Bedeutung*, of course), so does a sentence. The *Sinn* of a sentence is the thought (*Gedanke*) it expresses, and its *Bedeutung* is its truth-value, i.e. either the True or the False (one of Frege’s most controversial views).¹¹ Now, a judgment is, on the mental level, the acknowledgement of the truth of a thought, which is otherwise merely entertained; on the linguistic

⁸Notice however that a thorough characterization of Frege’s notion of judgment would require significantly more effort. Indeed, [3] is a book-length study on the topic, where Bell offers a critical analysis of Frege’s positions, in particular concerning assertion and the judgment stroke (Part III). Bell outlines many important issues and tension that emerge from Frege’s treatment of this notion, some of which echo the main objection to Frege’s notation being presented here.

⁹[6], p.34, note 7. This translation replaces the term “admission” in M. Black’s translation by “acknowledgement”, for the German “Anerkennung” (as suggested by P. Pagin [16]).

¹⁰[6], p.35. P. Pagin ([16], fn.3): “I have departed from Max Black by using the gerundives ‘judging’ and ‘advancing’ for the German ‘Urteilen’ and ‘Fortschreiten’, instead of ‘judgments’ and ‘advances’. I have also preferred ‘of its own kind’ to ‘peculiar’ for the German ‘einzigartiges’. Finally, Black has the indefinite ‘a truth value’ rather than the reflective ‘its truth value’, where the German is ‘seinem Wahrheitswert’.”

¹¹As pointed out by Bell ([3], p.86), the view that a sentence is the name of a truth-value is somewhat at odds with Frege’s insistence that the complete judgment stroke \vdash can only be appended to expressions that are propositional in character. But what emerges from *Über Sinn und Bedeutung* is that, concerning the sense/reference mechanism, there is no fundamental difference between single terms and sentences.

level, a judgment is the assertion (the expression of the belief) that the thought expressed by the sentence actually obtains. Thus seen, a judgment seems indeed to be a derivative entity, formed by the combination of a thought with the “acknowledgement of its truth”: judging is the act from a thought to a truth-value, and hence its product, the judgment, has a thought as one of its ingredients. That a thought may be seen as a more primitive entity also transpires from the observation that it can occur without the accompanying judgment, as is the case when one merely entertains a thought, but a judgment cannot occur without the accompanying thought.

This idea is, of course, what underlies Geach’s formulation of the so-called Frege point (what we here refer to as “judgment” is in Geach’s terminology “assertion”):

“A thought may have just the same content whether you assent to its truth or not; a proposition may occur in discourse now asserted, now unasserted, and yet be recognizably the same proposition [...] I shall call this point about assertion the Frege point”. ([11], p.449)

The general idea – present in several other influential theories such as Searle’s speech-act theory, among others – is that any speech-act is a combination of a particular force with a given content, and that speech-acts with different forces may nevertheless share the same content. For example, the following different speech-acts

“John, put beer in the fridge” (a command)

“John puts beer in the fridge” (an assertion)

“Does John put beer in the fridge?” (a question)

are presumed to have different forces but to share the same content, namely the content usually expressed by the that-clause “*that John puts beer in the fridge*”. Both Frege and Geach in fact focus on assertion, just as we here focus on judgment, thus disregarding other kinds of speech-acts;¹² but one of the strengths of the force-content distinction is that it allows for a neat, unified treatment of all kinds of speech-acts.

This account of speech-acts, and of judgments/assertions in particular, has not gone unchallenged,¹³ but it is still widely held to be correct. In particular, it emerges in many of Frege’s writings, in particular *The Thought*, and again, its most significant implication for the present purposes is the idea that a judgment is a compound entity (thus not a primitive entity), composed of a content (proposition, thought) and the judging force. But important though it is, my purpose here is not to offer a critical analysis of the force-content distinction as such, and at any rate not regarding philosophy of language and philosophy of mind.

¹²For the present purposes, there is no significant dissimilarity between assertions and judgments, except for the fact that judgments may be understood either as the purely mental operation or as its verbal manifestation, whereas assertions correspond exclusively to the latter. This fact was noticed by Frege himself ([8], p.62).

¹³Wittgenstein in the *Philosophical Investigations* is a prominent example. See ([15]) for further references.

As for logic, however, I have argued in the previous section that contents should not be the primitive elements of logic if one is to hold an epistemological view of the purpose of logic. In fact, contents seem to have a very small role to play within logic thus seen; but if this is so, then for the sake of theoretical parsimony, contents should not receive a particular form of representation in a logical language, or in any case this form of representation should not be a primitive element of the syntax. Now, this is not what one encounters in Frege's *Begriffsschrift*.

In par. 2 of the *Begriffsschrift* Frege introduces the symbols to be employed to represent judgment. He writes:

“A judgment will always be expressed by means of the sign

⊢

which stands to the left of the sign, or the combination of signs, indicating the content of the judgment.” ([10], p.11)

A few syntactic features of the use of ⊢ are already quite significant: ⊢ can neither be iterated nor occur embedded in a given expression, i.e. there is at most one occurrence of ⊢ per formula, all the way to its left. Hence, it is not possible to express something like: “I assert that I assert p to be the case” or “If I assert p to be the case, then I assert q to be the case”.

The vertical stroke is supposed to represent the act of judging the content to be the case, and accordingly is known as the “judgment stroke”; the horizontal stroke accounts for the fact that the content represented by a sign or signs is a combination of ideas and thus not a simple idea. A simple idea such as “house” is not apt to be judged as either true or false; only a combination of two or more ideas – say, “house” plus “large” – is judgeable. The horizontal stroke is thus the “content stroke”. It is patent that a representation of a judgment with this notation mirrors its threefold structure (as described above), involving (i) ideas; (ii) the combination thereof, yielding a thought; (iii) the advancing from a thought to a truth-value.

There is no point in disputing that Frege really held a threefold view of the structure of a judgment: the issue raised here is whether this structure must be fully represented for *the purposes of logical reasoning*. For Frege, the objects on which logic operates in order to draw inferences (the quintessential logical operation) are judgments, as is patent in the discussion on inference in par.6, and as discussed in the previous section. Accordingly, he proposes to represent the inference from a conditional judgment and the judgment of its antecedent to the judgment of its consequent as¹⁴

$$\frac{\begin{array}{c} \vdash B \rightarrow A \\ \vdash B \end{array}}{\vdash A}$$

Of course, the important thing to bear in mind is the fundamental difference between the occurrence of a content in a categorical judgment and its occurrence

¹⁴In the following I represent a conditional in the usual way here, instead of in Frege's two dimensional way, purely for typographical convenience (which only goes on to prove that Frege's notation is indeed not extremely practical).

in a hypothetical judgment of the form “If A, then B”. In the latter, neither the content expressed by the antecedent nor the content expressed by the consequent is asserted/judged to be the case; in both cases, one could say that these two thoughts are merely comprehended or entertained, and that the assertive force is exclusively borne by the logical connective in question. What is asserted is the relation between the two contents, not the contents themselves.

At the same time, this distinction also seems to entail that no occurrence of *modus ponens* (just to take a particularly familiar example) can ever be valid, as it would always constitute a fallacy of equivocation. If *modus ponens* is correctly expressed by the following formulation:

If p then q
 But p
 Thus q

then it would seem that both propositional variables p and q are used equivocally: in the first premise, they are used to indicate the unasserted corresponding contents, while in the second premise and conclusion they are used to indicate the assertion of these contents. The matter is not simply of notation: if the objects that the propositional variables stand for are not the same (assertive vs. unassertive contents), then one may go as far as wondering whether the argument actually goes through; as is well-known from syllogistic, the terms of the premises and of the conclusion of an argument must stand for the same things (respectively) in order for the argument to be valid. So the dilemma seems to be that either the things that the propositional variables stand for really are the same – in which case the second premise and the conclusion are redundant, that is if in the conditional premise the propositional variables already have assertive force – or they are not the same, in which case the argument will not go through on pain of equivocation. This is a real dilemma, but Frege seems to provide a solution to it that is as good as any in that it outlines the *difference* as well as the *sameness* of a categorical and a hypothetical occurrence of a content: on the one hand they are the same, as they share the same content, but on the other hand they are not the same, as they do not share the same force. Ultimately, this is an issue of propositional identity, as Frege is well aware of (*Begriffsschrift*, par.3).¹⁵

The point is though that, in order to emphasize the difference/sameness relation between the expressions of asserted and of unasserted content, one may either choose to take the asserted expression as the starting point on which a given operation is performed in order to obtain the unasserted expression, or one may choose to reverse the order of priority. Frege opted for the latter, and at least at first sight his choice seems quite felicitous: by “extracting” the assertive force from a judgment and representing it separately, the validity of *modus ponens* is guaranteed by the fact that the propositional variables do stand for the same things (respectively) in premises and conclusion, namely for contents. At the same time, it is clear that a different force is attached to them in their hypothetical occurrences in the first premise and in their categorical occurrences in the second premise and conclusion.

Now, in many (but not all) natural languages, the lack of assertive force in the embedded sentences of conditionals is represented by the use of a different

¹⁵This point is raised and discussed in [3], III.3.

verbal form, e.g. the subjunctive verbal moods in German, Latin, Portuguese, Italian and Spanish, to mention but a few examples. While we seem to do fine without such a distinction in our daily use of English,¹⁶ where subjunctive moods are hardly ever used (except for the occasional occurrence of phrases such as “*If I were you...*”), Frege could argue that, for logical and scientific purposes, it is of the utmost importance that this distinction be represented (cf. his critique of the imperfections of the “*Sprache des Lebens*” in the Preface of the *Begriffsschrift*). In Frege’s native German the use of non-indicative moods in non-assertive contexts (also in indirect speech) is very common, and Frege himself notes this connection between his judgment-stroke and the expressions of asserted and unasserted content in the “language of life” with different verbal moods – see ([3], ???).

However, the crucial point is that, in an artificial notation, a choice is to be made as to what is to count as the *canonical form* of the expression of a content: should we take the assertive expression of a content as the canonical form, and accordingly the non-assertive expression of it as a derivative form, or vice-versa?¹⁷ It is clear that in natural languages the assertive expression of a content, i.e. the expression of a judgment, is usually taken to be the canonical form, corresponding to the use of indicative verbal moods in such cases. By contrast, non-assertive expressions of content are (often, though not always) rendered in a derivative form, for example by means of subjunctive verbal moods, nominalizations or that-clauses, which are clearly derivative/non-canonical with respect to “plain” sentences with indicative verbs.

Frege, though, opted for the reverse solution: prompted perhaps by the notion of judgment as a compound of content and force (even though, of course, the *Begriffsschrift* antedates *Über Sinn und Bedeutung* by more than a decade), he took the expression of unasserted thoughts to be more primitive or canonical. The expression of a judgment is obtained by means of the application of the functor “judgment stroke” to the expression of an unasserted thought; the expression of a judgment is thus non-primitive and derivative in Frege’s notation.

While it may be motivated by a more general account of the “phenomenology” of judgments, this notational choice is in tension with Frege’s own insistence on the primacy of judgments for logic, insofar as judgments alone can advance knowledge. Notice that I certainly do not wish to imply that this tension is an outright contradiction or anything of the kind. All I am suggesting is that a different notational choice, one where the expression of judgments is assigned the role of canonical form and the expression of unasserted content is derived from the canonical form, might have been more in tune with the key role played by judgments, but not by contents, in Frege’s vision of logic. A non-canonical form could then be used whenever a non-assertive expression of content (e.g. embedded sentences in conditionals) is at stake.¹⁸

¹⁶But English features other devices to indicate a force other than the assertive force. For example, it features subject-verb inversion to signal the interrogative force.

¹⁷Here I assume that assertions/judgments are the chief speech-acts for logic. But the point could be expanded so as to include other kinds of speech-acts. The general question is whether the canonical form should correspond to the whole speech-act or to the content separated from the force.

¹⁸The challenge is though whether one can still obtain an operational logic with this notational choice, i.e. one that deals adequately with the modus ponens dilemma just discussed.

4 The aftermath

Frege's notational choice seems to me to have backfired: while he explicitly wanted judgments to be at the central stage, the fact that contents are represented as more basic than judgments may have suggested that one could simply do away with the “cumbersome” strokes for judgment and content. Indeed, to quite a few people they seem to have appeared as unnecessary philosophical subtleties with no serious import for logic: why not treat the logical relations between contents directly, instead of taking the detour via judgments and inferences? Isn't the judgment stroke redundant at best, but possibly even meaningless?¹⁹

In fact, Frege's notational choice concerning judgments seems to have been infelicitous for (at least) two reasons. Firstly, the bi-dimensional, non-linear representation of hypothetical judgments, in fact the notation of the *Begriffsschrift* as a whole, is hopelessly cumbersome and a type-setting nightmare. So the *Begriffsschrift* notation was virtually bound to die with Frege, especially after the appearance of the *Principia* notation, which is undoubtedly simpler, but whose simplicity in fact hides conceptual impoverishment. Secondly, as judgments are represented as derivative entities, one is led to wonder whether the whole judgment apparatus is really required for logical purposes. The generations following Frege seem by and large to have concluded that judgments are essentially superfluous for real logical analysis: gradually, judgments became almost entirely neglected. In effect, in 20th century logic judgments and inferences became “endangered species”, as the title of Sundholm's inaugural lecture in Leiden suggests ([17]). Rather than focusing on how an agent can attain new knowledge (justified judgments) on the basis of knowledge already available, a large portion of logicians became concerned exclusively with the logical relations between contents, and in particular with the relation of consequence (in the model-theoretic tradition). The meta-mathematical formalist tradition moved even further away from the pivotal role of judgment and inference for logic: in this tradition, well-formed formulas are no longer meaningful objects, and the goal is simply to study their properties and mutual relations, given a certain formalism. The formal languages typically used within the formalist tradition simply lack the resources to differentiate judgments from contents. Here is Sundholm's description of this phenomenon:

“Instead of propositions, formulas are used as formalistic substitutes. These well-formed formulas serve furthermore as end-formulas of the formal proof-trees, and are thus also the formalistic counterparts of judgments. From a formalistic point of view, there is no distinction between judgments and their contents. The sign ‘ \vdash ’, which was used as an assertion-indicator by Frege, Russell and Heyting, is in the metamathematical tradition a mathematical predicate of provability, which is to be applied to well-formed formulas.” ([17], p.13)²⁰

A few words on Whitehead and Russell's *Principia Mathematica* are in order here, as it seems to represent the beginning of the “fall”. The two authors saw

¹⁹Wittgenstein, *Tractatus*, prop. 4.442: “Frege's ‘judgment stroke’ \vdash is quite meaningless.”

²⁰My translation from the original Dutch text.

themselves as continuing Frege’s logicist program, in particular by salvaging Frege’s logic from paradox; but they made the conscious choice of not using Frege’s difficult notation. Rather, in terms of notation they acknowledge the foremost influence of Peano (*Principia*, p. 4). Presumably, however, they would have been committed to maintaining the most important conceptual features underlying Frege’s logical system, in particular the emphasis on judgments and inference.

Whitehead and Russell do maintain the sign of assertion as one of the (primitive) symbols in their logical language. But significantly, it is introduced (in p. 8) only *after* the introduction of the symbols standing for propositional contents (propositional letters), of the functional operations on contents (negation, disjunction etc.), and of truth-values. Remember that, in the *Begriffsschrift*, the notion of judgment is introduced in par.2, and only the general idea of using letters as symbols of generality precedes it (in par.1). Moreover, an embryonic state of the judgment-content conflation can be perceived in the remarks on inference immediately following the introduction of the sign of assertion (*Principia*, p. 9), more specifically on the inference from $\vdash p$ and $\vdash p \supset q$ to $\vdash q$:

“...for the sake of drawing attention to the inference which is being made, we shall write instead

$$\prime \vdash p \supset \prime q'$$

which is to be considered as a mere abbreviation of the threefold statement

$$\prime \vdash p' \text{ and } \prime \vdash p \supset q' \text{ and } \prime \vdash q'.$$

”

In this passage, a *propositional* operation, denoted by “ \supset ”, is suddenly applied to judgments; granted, Whitehead and Russell view it as a “mere abbreviation”, but even as an abbreviation it seems to indicate a categorical mistake, namely that of taking judgments and contents interchangeably.

The rest of the story of the exile of inference and judgment from logic has been told by Sundholm at different places – [17, 18, 20] – so we need not rehearse its details once again. The point that I wish to stress, however, is the role that Frege himself may have played in this development: by opting to represent contents, not judgments, in a canonical/primitive form, he may have induced the idea that contents, not judgments, are the primitive elements for logic. The first signs of the content/judgment conflation can already be perceived in Whitehead and Russell’s *Principia Mathematica*, as we have seen; from then on, the predominance of content/consequence over judgment/inference only became accentuated – with the exception of the few dissident voices who still insisted in the epistemological role of logic, i.e. precisely the intuitionist/constructivist tradition to which Sundholm belongs. In other words, perhaps the “logical atavism” advocated by Sundholm should have a slight revisionist component: Frege’s particular notational choices concerning judgments and contents might be in need of revision.

Of course, I am not claiming that Frege’s notational choices are the sole responsible for the exile of inference and judgment from logic; obviously, this

development is significantly more intricate, and certainly many other causes contributed for its unfolding. It seems to me that Frege’s notation is not even a major force behind the “ontological turn” in logic: but one cannot help but wonder what the outcome would have been, had Frege been more faithful to his own conceptual commitments in his notational choices. Would judgment and inference have stood at least a fairer chance? And could he have made a different choice, given his conceptual commitments? That he could (at least in theory) have made a different choice will be argued for in the next section.

But before we move on to the next section, there is one observation I would still like to make. In attributing priority to judgments over contents in Frege’s logical system, I am essentially relying in what he explicitly *says*, for example in the passages quoted in section 1 above. However, it may be argued that something else emerges from what he actually *does*, i.e. from the actual functioning of his logical machinery: it is conceivable that, in practice, the priority of contents over judgments is an integral, organic part of his logical system, *malgré* Frege himself. In this case, his notational choice would in fact be firmly grounded on a real albeit tacit aspect of the system and thus no coincidence at all. If this is so, then Frege’s responsibility in the subsequent expulsion of judgments from the realm of logic would go well beyond notational choices: by switching the focus entirely onto contents, later logicians might simply have been developing a veiled but crucial aspect of Frege’s own logic. Now, this issue cannot be adjudicated to a full extent at this point, but for now I leave it as a suggestion: it is at least theoretically possible that Frege did make the right notational choice after all, and that his conceptual insistence on judgments was in fact misguided and ultimately at odds with the actual machinery of his logical system. It would appear in any case that this (hypothetical) primacy of contents would not simply follow from the recognition of the force-content distinction (as I will argue below), but rather from other technical and structural aspects of Frege’s logic, which would have to be looked into in more detail at a different occasion.

5 Medieval representations of judgment and content

Frege’s representation of judgments as composed of contents and of assertive force may at first sight seem simply to follow from his “discovery” of propositionality (as captured in Geach’s “Frege point”). Once one recognizes that new propositional contents may be formed by means of propositional operations (typically: negation, conjunction, disjunction etc.) taking contents as arguments and yielding different contents as values, then it may seem that one has no choice but to represent contents as primitive elements of a logical system. To be sure, these operations do not take judgments as their arguments, nor do they yield new judgments: *inferences* are the operations that take judgments onto other judgments (and they are themselves judgments too). Of course, sometimes there is symmetry between propositional operations and inferences: from contents A and B one may form the content $A\&B$, just as from the judgments that A is the case and that B is the case one may (correctly) arrive at the judgment that A and B are the case. But this obviously does not hold irrespectively: from any two contents A and B one may form the content $A \rightarrow B$, but

the judgments that A is the case and that B is the case are in no way sufficient warrant for the judgment that $A \rightarrow B$ is the case.

Nonetheless, in this section I will argue that Frege’s theoretical assumptions and “discoveries” did not necessarily force upon him the particular notational choice that he eventually made, namely that of taking the representation of contents, not of judgments, as canonical/primitive. In order to motivate this claim, I will present a counterexample, i.e. an author who held views strikingly similar to Frege’s views, but who nevertheless made the reverse notational choice: the 12th century logician Peter Abelard.²¹

There are remarkable points of resemblance between the positions held by Frege (discussed so far) and the views held by Abelard. In fact, Christopher J. Martin credits Abelard for many of Frege’s most important “discoveries”, and claims that Abelard was an even greater logician than Frege.²² Of course, Frege made these discoveries independently, as he most certainly had no knowledge of Abelard’s ideas (in fact, most of Abelard’s legacy simply got lost already in the generations immediately following him). But Abelard’s logical starting point (what he had inherited from the previous traditions) was in fact extremely meager, while Frege could rely on much more elaborate material as his starting point, so in this sense Abelard’s accomplishment is more impressive.

For our purposes, there are three important points of similarity among Frege and Abelard:

- Both hold that a sentence (typically) expresses a content: a *dictum* for Abelard, a thought for Frege;
- Both recognize the force-content distinction and that different speech-acts (with different forces) can nevertheless express the same content (for Abelard, see [2] 3.05.13 and 3.01.101);
- Both identify propositional operations; they are thus both “inventors” of propositionality (for Abelard, see [13], section II.1).

In short, most of the points discussed so far that may have motivated Frege’s decision to represent contents instead of judgments as primitive entities in his logical notation are to be found in Abelard as well. Nevertheless, Abelard did *not* take the representation of contents, i.e. *dicta*, as the canonical linguistic form. Granted, Abelard is often rather careless in his use of language; in particular, he frequently uses important terminology inconsistently – see [13], p.166. This is of course a major flaw in a logician; in effect, a considerable amount of charity is needed when interpreting the linguistic expression of his ideas. But when charity is used, a solid and sophisticated logical system emerges.²³

²¹Technically, the choice in question was not really “made” by Abelard, as he was mostly following conventions and usages already established. But the point to be made here is simply that propositionality and force-content distinction are perfectly compatible with a notational choice taking the expression of assertions as the canonical form.

²²Personal conversation.

²³Another point of similarity relating Frege to Abelard is that Abelard’s logical system was also proven to be ultimately untenable, just as Frege’s system was shown to be ultimately paradoxical. Alberic of Paris seems to have put forward a killing argument against Abelard’s logic, showing that it allows for the derivation of “If Socrates is human and Socrates is not an animal, then it is not the case that Socrates is human and Socrates is not an animal” – a very embarrassing conclusion indeed. In C.J. Martin’s terms ([13], p.191): “Abelard’s various intuitions about the propositional connectives cannot be reconciled”. But these embarrassments do not affect the present analysis.

Before we inspect Abelard’s linguistic devices for representing judgments and contents, a few words on his famous *dicta* (contents) are required. A *dictum* is what an assertion says to be the case; in Abelard’s formulation, “*that which is said by a proposition [assertion]*” ([2], 3.04.26). Thus, a *dictum* is very much like a Fregean *Sinn/Gedanke*, insofar as a *Sinn* is what is expressed by a locution. But unlike Frege, Abelard resists any kind of reification of *dicta*: even though a sentence says something, one is however not entitled to conclude that there is some *thing* that is said by a sentence. *Dicta* have thus a delicate, if not to say problematic, ontological status (see [12], section IV.4).

For our purposes, what matters is how contents (i.e. *dicta*) and judgments are represented. If the points of similarity between Frege and Abelard raised above would indeed necessitate taking the representation of contents as canonical and the representation of judgments as derivative, then one would expect Abelard to follow a similar notational convention. But this is not what one finds in his writings. Abelard represents what we here refer to as judgments (in his terminology, *enuntiatio* or *propositio*) by canonical sentences of the nominative-subject/ indicative-verb form; here, he is simply following common usage, so this is no great feat. More significant, though, are the devices he uses to refer to *dicta*; indeed, *dicta* are represented by means of constructions *derived* from the canonical form of sentences described above. The two typical constructions to refer to *dicta* are the accusative-subject/ infinitive-verb construction and the that(*quod*)-clause construction. Both are particularly conspicuous in two passages where he discusses the notion of truth,²⁴ namely [2] 3.01.100 and 3.04.26. The formulations used by Abelard are “*Verum est Socratem sedere*” and “*Verum est Socratem esse hominem*”, and the nominalized clauses are seen as indicating *dicta* (even though, properly speaking, a *dictum* is not a thing).

One might expect that similar constructions would be used for the occurrence of unasserted contents in conditionals. Abelard does recognize that, in a true conditional such as “If Socrates is a pearl, then Socrates is a stone”,²⁵ the whole is (correctly) asserted while neither the antecedent nor the consequent is asserted (*proponitur*) ([1], 90) – if they were asserted, then the asserted conditional would be false, as both are false. But the conditional is true, which plainly shows that the two embedded sentences are not asserted even though the whole is. But perhaps disappointingly, the two embedded sentences are represented in the canonical nominative-indicative form, i.e. the same form that is typically used to represent asserted contents: “*Si Socrates est margarita, Socrates est lapis*”. But of course, in a conditional, the two relata are not assertions but contents. In Abelard’s defense, one might say that this was simply the linguistic convention to formulate such conditionals at the time; but ideally, in a language where the judgment-content distinction is consistently represented, the embedded sentences in a conditional should be formulated with the appropriate convention for expressing contents.

Be that as it may, and even though Abelard is not entirely consistent in his representations of judgments (assertions) and contents (*dicta*), it seems to me that a lesson can still be learned from the common points between Frege

²⁴For Abelard, assertions (*propositiones*) are not the actual bearers of truth and falsity, properly speaking. Rather, these bearers are the *dicta*, which are true when they describe things as they are (*ita est in re*) and false otherwise. Assertions are only derivatively true or false, i.e. when they “say” a *dictum* that is true or false.

²⁵At the time, it was believed that a pearl was a stone.

and Abelard and their divergence concerning the choice of the canonical form of representing judgments or contents. As we have seen, Frege chooses contents to be represented canonically, i.e. as primitive elements of the syntax, and the representation of judgments is obtained from the representation of contents by means of the application of a functor, the judgment stroke. Abelard, by contrast, represents judgments by means of a canonical form; contents are in turn represented by constructions obtained from this canonical form, namely the accusative-plus-infinitive construction or that-clauses. This approach suggests that contents are obtained from judgments, whereas Frege's approach suggests that judgments are obtained from contents. The upshot is thus that Frege's views on the force-content distinction and on propositionality do not seem to force a particular choice for the representation of contents and judgments. In any case, taking only these elements into account,²⁶ it seems that Frege could just as well have taken the representation of judgments as a primitive element of his syntax, instead of that of contents; this would in any case have been more attuned to the primacy he explicitly attributes to judgments in logic. More generally: had he done so, judgments and inferences might have had a better future in subsequent logical developments.

6 Conclusion

Generally, I take it to be important that a given logical notation emulate for as much as possible the conceptual structure underlying the logical system it represents. There is however a palpable tension in Frege's notational choice for representing judgments and contents and his epistemological view of logic. The comparison with Abelard was intended to suggest that, at least from a purely conceptual point of view, propositionality and the force-content distinction are not incompatible with a notation where the expression of judgments, not of contents, takes a canonical form.

Naturally, there are several fundamental differences between Abelard and Frege and their respective logical systems. The most important of them seems to be the fact that Frege explicitly introduces a formal language, with artificially created symbols, whereas Abelard expresses his ideas in the academic Latin of the 12th century. In fact, academic Latin was to become much more regimented in the subsequent centuries (in particular in the 14th century onwards), and with Abelard these conventions were still in their germinal state. Related to this is the fact that Frege's logic allows for a much higher level of abstraction. But precisely because Frege stipulated a new logical language, he could in principle simply have given a canonical form to the representation of judgments. So either this choice was somehow arbitrary, or there are hidden elements in his logic forcing the particular choice he actually made.

The challenge is of course to design a logical system/notation that has judgments represented canonically, as primitive elements, but which still functions appropriately for the purposes of logical reasoning and manipulation. Such a

²⁶As suggested above, there might be other aspects of Frege's logical system that do necessitate this particular choice. The comparison with Abelard seems in any case to indicate that one would have to look further for these aspects, as propositionality and the force-content distinction alone seem not to be sufficient to force the choice of a canonical representation of contents instead of judgments.

system would then be truly faithful to the tenets of the epistemological view of logic. But it would, for example, have to offer a satisfactory solution to the *modus ponens* dilemma discussed above, and that seems *prima facie* quite a challenge. Suppose for instance that in such a language judgments are represented by capital letters (A, B, C, \dots) and that contents are represented derivatively, i.e. their representations are obtained by means of an operation on judgments, for example: $\underline{A}, \underline{B}, \underline{C}$ (the underlying representing the content-forming operation starting from judgments). In such a notation, some usual rules for the logical connectives would be:

$$\frac{\underline{A} \rightarrow \underline{B} \quad A}{B} \quad \frac{\underline{A \& B}}{A} \quad \frac{A}{\underline{A} \vee \underline{B}}$$

Now, isn't this a little bit odd? How can A be applied to $\underline{A} \rightarrow \underline{B}$ in order to obtain B , if neither A nor B (but only the corresponding contents) actually occur in the first premise? Could this be a categorical mistake? From this perspective, Frege's solution appears as rather ingenious indeed.

In modern systems that do take judgments seriously, such as Martin-Löf's Constructive Type-Theory (CTT, which is, as we know, an important source of inspiration for Sundholm), judgments are syntactically construed on the basis of the expressions of contents (propositions). A typical judgment in CTT is $a : A$, which is the assertion to the effect that the object that a stands for (a proof, a construction) is of *type* A , which may mean for example that it is a proof of the proposition (content) that A stands for. It seems to me that this sort of type-assertion is indeed a "pure" assertion, not a manipulation of contents in disguise. However, (overtly inspired by Frege's judgment stroke) Martin-Löf also represents an assertion of content A as A *true* (see for example [14]). Thus, while his analysis of the concepts of judgment and content are extremely lucid and sophisticated, it appears to me that Martin-Löf does perpetuate at least some of Frege's notational choices, and thus still does not deliver a notational system that does full justice to the intuition that judgments are the basic elements for logic (even though he is undoubtedly the best contender in this respect).

But what if the endeavor of designing a notation where judgments are truly primary proves to fail or to be exceedingly difficult? What if a logical system/notation where judgments are represented canonically and contents are represented derivatively turns out to be significantly less manageable (if at all manageable) than systems/notations where the reverse occurs? If this turns out to be the case, this outcome may suggest two conclusions concerning the epistemological vision of logic:

1. A radical conclusion: the vision of logic as an epistemological tool is fundamentally misguided. Logic is not about judgments and inferences; it is about contents and consequences.
2. A moderate conclusion: the emphasis on judgments and inferences belongs to the *philosophy* of logic, to what we think we are doing when doing logic, but not to logic itself, i.e. to the general principles and operations characterizing the enterprise as such.

In the spirit of conclusion 1), it may be added that Frege's system is a somewhat awkward hybrid being, combining propositionality insights with an

old-fashioned conception of the purpose of logic, which are nevertheless in tension with each other. In particular, his choice to represent contents instead of judgments canonically would belong to the insightful and successful part of his program. No wonder that in subsequent developments judgments and inferences became endangered species: they simply were not in their natural habitat in the first place, and thus had not much of a chance of survival.

Conclusion 1) would undoubtedly enrage Sundholm, Frege and the partisans of the epistemological view of logic in general. Still, it seems to me that they owe us an explanation of the fact that, even in the different logical systems that emphasize the role of judgments and inferences mentioned in this paper (Frege's in particular but also CTT), contents – not judgments – are represented canonically. Aren't contents then the most basic logical entities after all?

Conclusion 2) does not challenge the propriety of the epistemological view of logic as a whole, but relegates the importance of judgments and inferences to the realm of what we do with, and think about, logic. On this view, we may choose to give logic the epistemological import that Sundholm and Frege attribute to it, but we need not do so, as it is not an inherent feature of logic as such. Logic can be applied to the production of new knowledge, but in truth it essentially concerns relations of incompatibility and truth-preservation between contents (and ultimately between the states-of-affairs that contents correspond to). When we intend logic to be used as an epistemological tool, then we are indeed concerned with the assertion of these contents, with judgments, but otherwise logic does just fine without them.

But of course, these are conditional conclusions: *if* it is impossible, too difficult, or just considerably more difficult to work with a notation having judgments as primitive elements, this *might* indicate that the epistemological view of logic is “less natural” after all. I of course do not exclude the possibility of a successful judgment-based notation, but I must say that I am unconvinced by those that have been proposed so far. Moreover, issues such as those underlying the *modus ponens* dilemma discussed above seem to me to be real challenges for such a judgment-based notation. Thus, although I am conceptually very sympathetic to the epistemological view of logic, I am somehow discouraged by the difficulties devising a notation faithful to its basic tenets, which makes me wonder whether logic is not after all essentially about contents and the states-of-affair that they represent.

Either way, it seems to me that the canonical form of representation attributed to contents in logical systems that nevertheless emphasize the role of judgments, such as Frege's, is something in want of an explanation. It is in any case something to be pondered on, and which may give us further insights in the debate opposing the epistemological and the ontological views of logic.

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